

COMPACTIFIED CONFIGURATION SPACE OF POINTS ON A LINE AND HOMOTOPIES OF A_∞ MORPHISMS

THEO BACKMAN

Abstract

We construct a configuration space model for a particular 2-colored differential graded operad encoding the structure of two A_∞ algebras with two A_∞ morphisms and a homotopy between the morphisms. The cohomology of this operad is shown to be the well-known 2-colored operad encoding the structure of two associative algebras and of an associative algebra morphism between them.

0. INTRODUCTION

Many important algebraic operads can be reinterpreted as operads of chains on a topological operad. One of the first and most important operads is the topological operad of little disks. The associated *chain* operad of little disks was studied in a paper by Cohen [Co], where he proved that its homology operad coincides with the operad of *Gerstenhaber algebras*. Another example is given by the operad of little intervals $D_1(\mathbb{R})$. The n -th part of this operad is (roughly speaking) given by the space of embeddings of n copies of \mathbb{R} into \mathbb{R} such that the image intervals are disjoint. The representations of $D_1(\mathbb{R})$ are the same thing as A_∞ spaces and the chains of this topological operad is a differential graded operad that is quasi-isomorphic to the operad \mathcal{A}_∞ . There is however another very useful way to connect the theory of A_∞ algebras to the theory of geometric operads.

Consider n points on the real line modulo the action of the affine group; $x \mapsto \lambda x + c$ where $\lambda \in \mathbb{R}^+$ and $c \in \mathbb{R}$. The space of such configurations of points is an $n - 2$ dimensional manifold $C_n(\mathbb{R})$. This manifold $C_n(\mathbb{R})$ can be suitably compactified into a closed manifold with corners $\overline{C}_n(\mathbb{R})$ in a such a way that the whole family $\{\overline{C}_n(\mathbb{R})\}_{n \geq 2}$ gives us an operad in the category of smooth manifolds with corners. The associate operad of fundamental chains is identical to the operad of \mathcal{A}_∞ algebras. Note that in this approach we get a geometric interpretation of the \mathcal{A}_∞ operad in terms of manifolds - *not just topological spaces!* Therefore this approach gives us new mathematical tools when studying strongly homotopy algebras, as for example, manifolds with corners are always equipped with sheafs of differential forms which one can integrate and which obey the Stokes theorem. Therefore such an interpretation of an algebraic operad in terms of an operad of configuration spaces opens up the possibility of obtaining transcendental results; those results that cannot be achieved just through homological algebra and perturbative methods. There are two such famous transcendental results due to Kontsevich.

In the 90s Kontsevich made a ground breaking contribution to the field of mathematical physics by proving his *Formality conjecture* [Ko1]. The result gives an L_∞ quasi-isomorphism

$$\mathcal{K} : (\mathcal{T}_{poly}(\mathbb{R}^d), [-, -]_{SN}, d = 0) \longrightarrow (\mathcal{D}_{poly}(\mathcal{O}(\mathbb{R}^d)), [-, -]_G, d_H)$$

from the Lie algebra of polyvectorfields on \mathbb{R}^d equipped with the Schouten-Nijenhuis bracket and the trivial differential to the Lie algebra of polydifferential operators on smooth functions on \mathbb{R}^d equipped with the Gerstenhaber bracket and the Hochschild differential. The proof is via an explicit construction of the map given by integrals on configuration spaces. Kontsevich formality theorem can be formulated as a morphism of operads, i.e. as a morphism from the operad of fundamental chains of Kontsevich configuration spaces to the operad of Kontsevich graphs.

The second such transcendental result due to Kontsevich gives an explicit proof of the formality of the little disk operad [Ko2].

The 2-colored operad $\mathcal{Mor}(As)_\infty$ also admits a nice configuration space model [Me]. In this case one studies compactifications of configurations of n points on the line modulo just translation. The major achievement of this paper is a construction of a configurations space model for the 2-colored dg operad $\mathcal{Ho}(As)_\infty$ which controls a pair of A_∞ algebras, (V, μ^V) , (W, μ^W) , a pair of A_∞ morphism between them $f, g : (V, \mu^V) \rightarrow (W, \mu^W)$ and a homotopy between these morphism,

$$h : f \sim g.$$

Put another way a representation of our 2-colored operad is a diagram in the category of A_∞ algebras like this:

$$\begin{array}{ccc}
& f & \\
(V, \mu^V) & \begin{array}{c} \xrightarrow{\quad} \\ \parallel h \\ \xrightarrow{\quad} \end{array} & (W, \mu^W) \\
& g &
\end{array}$$

This time we study a suitable compactification of n different points on a line (without taking a quotient with respect the action of any Lie group). In this construction we recover the earlier configuration space models for A_∞ algebras and A_∞ morphisms which we discussed above. We also calculate the cohomology of the 2-colored dg operad $\mathcal{H}o(As)_\infty$ and show that it is equal to $\mathcal{M}or(As)$. This result proves that $\mathcal{H}o(As)_\infty$ is a non-minimal model of $\mathcal{M}or(As)$. This result also implies, after some additional work, that $\mathcal{H}o(As)_\infty$ is a non-minimal quasi-free model of the 2-colored dg operad $\mathcal{H}o(As)$ which is, by definition, the operad encoding the structure of two dg associative algebras, two algebra morphisms between them and a homotopy between these two morphisms.

0.1. Outline. The paper is divided into three sections.

In section 1 an exposition of some basic facts of algebra, and specifically A_∞ algebras, are given. We will describe a sort of translation between two equivalent definitions of A_∞ algebras and related notions.

In section 3 we will introduce two families of configuration spaces, describe how they can be compactified to form operads of topological spaces and describe how their associated face complexes are related to A_∞ algebras.

In section 4 some novel results are given. We describe a compactification of the family of configuration space of n points on a line. This space is described to be an operad of smooth manifolds with corners. The face complex is identical to the two-colored operad of homotopies between a pair of A_∞ morphisms, $\mathcal{H}o(As)_\infty$. We also determine the cohomology of $\mathcal{H}o(As)_\infty$ and prove that it is $\mathcal{M}or(As)$

1. COALGEBRAS AND A_∞ ALGEBRAS

Through out this text we let k be a field of characteristic zero.

Definition 1.1. A coassociative graded coalgebra C over the field k is a graded k -vector space with a degree zero comultiplication map $\Delta : C \rightarrow C \otimes C$ such that the following diagram commutes:

$$\begin{array}{ccc}
C \otimes C \otimes C & \xleftarrow{\text{id} \otimes \Delta} & C \otimes C \\
\uparrow \Delta \otimes \text{id} & & \uparrow \Delta \\
C \otimes C & \xleftarrow{\Delta} & C
\end{array}$$

We say that a coassociative coalgebra is counital if there exist a map $\eta : k \leftarrow C$, called the counit, such that the following diagram commutes:

$$\begin{array}{ccccc}
k \otimes C & \xleftarrow{\eta \otimes \text{id}} & C \otimes C & & \\
& \searrow \cong & \swarrow \Delta & & \\
& & C & & \\
& \swarrow \cong & \nwarrow \Delta & & \\
k \otimes C & \xleftarrow{\text{id} \otimes \eta} & C \otimes C & &
\end{array}$$

Definition 1.2. Let (C, Δ_C, η_C) and (D, Δ_D, η_D) be coalgebras. A map of coalgebras $F : C \rightarrow D$ is a k -linear map such that the following diagrams commutes

$$\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\downarrow \Delta_C & & \downarrow \Delta_D \\
C \otimes C & \xrightarrow{F \otimes F} & D \otimes D
\end{array}
\quad
\begin{array}{ccc}
k & \xleftarrow{\eta_C} & C \\
\downarrow \text{id} & & \downarrow F \\
k & \xleftarrow{\eta_D} & D
\end{array}$$

Definition 1.3. Let C be a graded coalgebra. A linear map $b \in \text{Hom}^n(C, C)$ such that the following diagram commutes

$$\begin{array}{ccc}
C & \xrightarrow{b} & C \\
\downarrow \Delta & & \downarrow \Delta_C \\
C \otimes C & \xleftarrow{\text{id} \otimes b + b \otimes \text{id}} & C \otimes C
\end{array}$$

Is called a degree n coderivation. The set of degree n coderivations of a coalgebra C form a k -vector space and is denoted $\text{CoDer}^n(C)$.

Remark 1.4. A degree one coderivation b such that $b^2 = b \circ b = 0$ is called a codifferential.

Definition 1.5. The pair (C, b) where C is a graded coalgebra and b codifferential of C is called a differential graded coalgebra, or just dg coalgebra.

Morphism of dg coalgebras are morphism of graded coalgebras which commute with codifferentials.

Definition 1.6. Tensor coalgebras. Let V be a graded vector space. The tensor coalgebra $T_c V$ is as a vector space the direct sum $\bigoplus_{k \geq 0} V^{\otimes k}$, where $V^{\otimes i}$ is the i -times iterated tensor product with itself,

$$V^{\otimes i} = \underbrace{V \otimes \dots \otimes V}_{i\text{-times}}.$$

$T_c V$ can be given a coalgebra structure with the coproduct map

$$\Delta : T_c V \rightarrow T_c V \otimes T_c V$$

given on summand $T_c^n V = V^{\otimes n}$ as

$$\Delta : (v_1, \dots, v_n) \rightarrow \sum_{i=0}^n (v_1, \dots, v_i) \otimes (v_{i+1}, \dots, v_n),$$

where the term for $i = 0, n$ are $1 \otimes (v_1, \dots, v_n)$ and $(v_1, \dots, v_n) \otimes 1$ inside $V^{\otimes 0} \otimes V^{\otimes n}$ and $V^{\otimes n} \otimes V^{\otimes 0}$, respectively.

The reduced tensor coalgebra $\overline{T}_c V$ is as a vector space the direct sum $\bigoplus_{i \geq 1} V^{\otimes i}$, with coproduct

$$\Delta : \overline{T}_c V \rightarrow \overline{T}_c V \otimes \overline{T}_c V$$

given, as above, on summands as

$$\Delta : (v_1, \dots, v_n) \rightarrow \sum_{i=1}^{n-1} (v_1 \dots v_i) \otimes (v_{i+1}, \dots, v_n).$$

Remark 1.7. From the coproduct we define the partial coproducts:

$$\Delta_{a+b}^{a,b} := V^{\otimes(a+b)} \hookrightarrow \overline{T}_c V \xrightarrow{\Delta} \overline{T}_c V \otimes \overline{T}_c V \rightarrow V^{\otimes a} \otimes V^{\otimes b}.$$

This will be a convenient short hand in many of the proofs of this section.

Proposition 1.8. A map of vector spaces $b : \overline{T}_c V \rightarrow V$ can be lifted to a unique coderivation of coalgebras,

$$B : \overline{T}_c V \rightarrow \overline{T}_c V,$$

such that $pr_1 \circ B = b$, when pr_1 is the natural projection $\overline{T}_c V \rightarrow V$. If B_n^m denotes the composition

$$B_n^m : V^{\otimes n} \hookrightarrow \overline{T}_c V \xrightarrow{B} \overline{T}_c V \rightarrow V^{\otimes m},$$

then the explicit formula for B_n^m is given by

$$B_n^m = \begin{cases} 0 & \text{if } n < m \\ \sum_{i+j=m-1} \text{Id}^{\otimes i} \otimes b_{n+1-m} \otimes \text{Id}^{\otimes j} & \text{if } n \geq m \end{cases}$$

where $b_a := b|_{V^{\otimes a}}$. Furthermore, the map B is recovered as a product;

$$B = \prod_{n \geq 1} B^n \quad B^n = \prod_{m \geq 1} B_m^n,$$

and note that $B_a^1 = b_a$

Proof. The proof is by induction. The case B_n^1 is clear from the projection property. Assume that for all $m < M$ we have that B_n^m is given by the formula. The equation

$$\Delta B = (\text{Id} \otimes B + B \otimes \text{Id}) \Delta$$

is true. Specifically we can restrict its input to be $V^{\otimes n}$ and its output to be in $V^{\otimes(M-1)} \otimes V$, in which case the formula becomes:

$$\Delta_M^{M-1,1} B_m^M = (\text{Id}^{\otimes(M-1)} \otimes B_{n+1-M}^1 + B_{n-1}^{M-1} \otimes \text{Id}) \Delta_n^{n-1,1}.$$

By the induction hypothesis we know that $B_{n-1}^{M-1} = \sum_{i+j=M-2} \text{Id}^{\otimes i} \otimes b_{n+1-M} \otimes \text{Id}^{\otimes j}$ so as an M -tensor the right hand side has the desired form; it's the formula given for B_n^M . \square

Proposition 1.9. *A map of vector spaces $f : \overline{T}_c V \rightarrow W$ can uniquely be lifted to a morphism of coalgebras $F : \overline{T}_c V \rightarrow \overline{T}_c W$, such that $pr_1 \circ F = f$, when pr_1 is the natural projection $\overline{T}_c W \rightarrow W$. If we let F_n^m be a composition;*

$$F_n^m : V^{\otimes n} \hookrightarrow \overline{T}_c V \xrightarrow{F} \overline{T}_c W \rightarrow W^{\otimes m}$$

and let $f_k = f|_{V^{\otimes k}}$ then, explicitly, F_n^m will be of the form

$$F_n^m = \begin{cases} 0 & \text{if } n < m \\ \sum_{i_1 + \dots + i_m = n} f_{i_1} \otimes \dots \otimes f_{i_m} \end{cases}$$

where F can be recovered as the product;

$$F = \prod_{m \geq 1} F^m \quad F^m = \prod_{n \geq 1} F_n^m.$$

Proof. By the property of the projection, $pr_1 \circ F = f$, it follows that $F_n^1 = f_n$. We proceed by induction; assume that

$$F_n^m = \sum_{i_1 + \dots + i_m = n} f_{i_1} \otimes \dots \otimes f_{i_m},$$

for all $m < M$. The equation

$$(F \otimes F) \circ \Delta = \Delta \circ F$$

can be restricted to taking the input $V^{\otimes n}$ and having the output $W^{\otimes(M-1)} \otimes W$, in which case it becomes

$$\Delta_M^{M-1,1} F_n^M = \sum_{i+j=n} (F_i^{M-1} \otimes F_j^1) \Delta_n^{i,j}.$$

Now we can expand F_i^{M-1} with the induction hypothesis and compare the two sides of the equation as an M -tensor. It follows that F_n^M is given by the formula in the description of the theorem. \square

Proposition 1.10. *Let $F, G : \overline{T}_c V \rightarrow \overline{T}_c W$ be two morphisms of coalgebras and let $h : \overline{T}_c V \rightarrow W$ be a map of vector spaces then, there exist a unique map $H : \overline{T}_c V \rightarrow \overline{T}_c W$, such that $(H \otimes G + F \otimes H) \Delta = \Delta H$ and so that $pr_1 \circ H = h$ when pr_1 is the natural projection $\overline{T}_c W \rightarrow W$. Define H_n^m as the composition*

$$H_n^m : V^{\otimes n} \hookrightarrow \overline{T}_c V \xrightarrow{H} \overline{T}_c W \rightarrow W^{\otimes m}$$

Explicitly H_n^m is of the following form

$$H_n^m = \begin{cases} 0 & \text{if } m > n \\ \sum_{a+b=m-1} \sum_{i_1 + \dots + i_a + s + j_1 + \dots + j_b = n} F_{i_1}^1 \otimes \dots \otimes F_{i_a}^1 \otimes h_s \otimes G_{j_1}^1 \otimes \dots \otimes G_{j_b}^1, \end{cases}$$

where $a, b \geq 0$, $s > 0$ and, F_k^1 and G_l^1 are as in the previous theorem, and $h_i := h|_{V^{\otimes i}}$. From H_n^m we can recover H by taking the product;

$$H = \prod_{m \geq 1} H^m \quad H^m = \prod_{n \geq 1} H_n^m.$$

Proof. We prove this with induction. When $m = 1$ this follows from the projection property; $H_n^1 = h_n$. Assume that

$$H_n^m = \sum_{a+b=m-1} \sum_{i_1 + \dots + i_a + s + j_1 + \dots + j_b = n} F_{i_1}^1 \otimes \dots \otimes F_{i_a}^1 \otimes h_s \otimes G_{j_1}^1 \otimes \dots \otimes G_{j_b}^1$$

for all $m < M$. Restrict the input to $V^{\otimes n}$ and consider the projection to the $(M-1, 1)$ -th component in the equation $(H \otimes G + F \otimes H) \Delta = \Delta H$ to get:

$$\Delta_M^{M-1,1} \circ H_n^M = \sum_{i+j=n} (F_i^{M-1} \otimes H_j^1 + H_i^{M-1} \otimes G_j^1) \circ \Delta_n^{i,j},$$

if we expand H_i^{M-1} with the induction hypothesis and F_i^{M-1} with the previous theorem then we see that this, as an M -tensor, is precisely the formula that the theorem predicts. \square

Definition 1.11. *An A_∞ algebra is a graded vector space V equipped with the structure of a codifferential b_V on the associated reduced tensor coalgebra (of the shifted vector space);*

$$b_V : \overline{T}_c sV \rightarrow \overline{T}_c sV.$$

A morphism of A_∞ algebras

$$f : (V, b_V) \rightarrow (W, b_W)$$

is a morphism of dg coalgebras

$$F : \overline{T}_c sV \rightarrow \overline{T}_c sW.$$

Let F and G be the morphism of dg coalgebras

$$F, G : (\overline{T}_c sV, b_V) \rightarrow (\overline{T}_c sW, b_W),$$

where b_V and b_W are codifferentials giving V and W the structure of A_∞ algebras. We say that a map

$$H : \overline{T}_c sV \rightarrow \overline{T}_c sW$$

is a homotopy of F and G if it satisfies two relations:

- (1) $(F \otimes H + H \otimes G) \circ \Delta_V = \Delta_W \circ H$
- (2) $F - G = b_W \circ H + H \circ b_V$.

Proposition 1.8, 1.9 and 1.10 provides us with a way to reinterpret the definitions concerning A_∞ algebras without referencing the tensor coalgebra explicitly.

Theorem 1.12. *An A_∞ algebra structure on the graded vector space V is a sequence of maps $m_n : V^{\otimes n} \rightarrow V$ of degree $2 - n$ such that following equations are satisfied*

$$\begin{aligned} m_1 \circ m_1 &= 0 \\ -m_2 \circ (\text{Id} \otimes m_1) - m_2 \circ (m_1 \otimes \text{Id}) + m_1 \circ m_2 &= 0 \\ -m_2 \circ (m_2 \otimes \text{Id}) - m_2 \circ (\text{Id} \otimes m_2) + m_3 \circ (m_1 \otimes \text{Id}^{\otimes 2}) \\ + m_3 \circ (\text{Id} \otimes m_1 \otimes \text{Id}) + m_3 \circ (\text{Id}^{\otimes 2} \otimes m_1) + m_1 \circ m_3 &= 0 \\ &\vdots \\ \sum_{s+j+t=n} (-1)^{s+jt} m_{s+1+t} \circ (\text{Id}^{\otimes s} \otimes m_j \otimes \text{Id}^{\otimes t}) &= 0 \end{aligned}$$

Proof. The proof is a matter of expanding the expression $b_V \circ b_V = 0$ with proposition 1.8 and recognizing that $m_n = s^{-1} \circ b_V^n \circ s^{\otimes n}$, where b_V^n is the restriction of b_V to $(sV)^{\otimes n}$ followed by the projection onto sV . The sign factor comes from applying the Koszul sign rule when shifts are reorganized. \square

We will occasionally denote an A_∞ algebra with the pair (V, m^V) , where m^V is the system of maps given in the above theorem.

Theorem 1.13. *A morphism of A_∞ algebras $f : (V, m^V) \rightarrow (W, m^W)$ is a collection of maps $f_n : V^{\otimes n} \rightarrow W$ of degree $1 - n$ such that*

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t} \circ (\text{Id}^{\otimes r} \otimes m_s^V \otimes \text{Id}^{\otimes t}) = \sum_{q=1}^n \sum_{i_1+\dots+i_q=n} (-1)^p m_q^W \circ (f_{i_1} \otimes \dots \otimes f_{i_q})$$

where $p = (q-1)(i_1-1) + (q-2)(i_2-1) + \dots + 2(i_{q-2}-1) + (i_{q-1}-1)$.

Proof. Let $F : (\overline{T}_c sV, B_V) \rightarrow (\overline{T}_c sW, B_W)$ be a coalgebra morphism. Explicitly f_n is given as $s^{-1} \circ F_n^1 \circ s^{\otimes n}$, where F_n^1 is the restriction of F to the n :th component followed by the projection to the first; $F_n^1 : (sV)^{\otimes n} \rightarrow sW$.

We start with the equation $B_W \circ F = F \circ B_V$. In it we restrict the input to $(sV)^{\otimes n}$ and output to sW . The result is

$$\sum_{i=1}^n (B_W)_i^1 \circ F_n^i = \sum_{j=1}^n F_j^1 \circ (B_V)_n^j.$$

By proposition 1.8 and 1.9 $B_n^m = \sum_{i+j=m-1} \text{Id}^{\otimes i} \otimes B_s^1 \otimes \text{Id}^{\otimes j}$ and $F_n^m = \sum_{n_1+\dots+n_m=n} F_{n_1}^1 \otimes \dots \otimes F_{n_m}^1$. Using these explicit formulas we arrive at the expression

$$\sum_{i=1}^n (B_W)_i^1 \circ \left(\sum_{n_1+\dots+n_i=n} F_{n_1}^1 \otimes \dots \otimes F_{n_i}^1 \right) = \sum_{i=1}^n F_i^1 \circ \left(\sum_{a+b=i-1} \text{Id}^{\otimes a} (B_V)_{n+1-i}^1 \otimes \text{Id}^{\otimes b} \right)$$

\square

Theorem 1.14. *Let (V, μ^V) and (W, μ^W) be two A_∞ algebras and let $f, g : (V, \mu^V) \rightarrow (W, \mu^W)$ be two A_∞ algebra morphisms given on the form of maps*

$$\begin{aligned} f_n &: V^{\otimes n} \rightarrow W \\ g_n &: V^{\otimes n} \rightarrow W \end{aligned}$$

A system of maps of graded vector spaces $h_n : V^{\otimes n} \rightarrow W$ of degree $-n$ is a homotopy of f and g if

$$\begin{aligned} f_n - g_n &= \sum_{m=1}^n \sum_{\substack{k+l=m-1 \\ i_1+\dots+i_k+t+j_1+\dots+j_l=n}} (-1)^s \mu_m^W \circ (f_{i_1} \otimes \dots \otimes f_{i_k} \otimes h_t \otimes g_{j_1} \otimes \dots \otimes g_{j_l}) \\ &+ \sum_{i+j+k=n} (-1)^{ij+k} h_{i+1+k} \circ (\text{Id}^{\otimes i} \otimes \mu_j^V \otimes \text{Id}^{\otimes k}), \\ s &= l + \sum_{1 \leq a \leq l} (1 - j_a)(n - \sum_{b \geq a} j_b) + t \sum_{1 \leq a \leq k} i_a + \sum_{2 \leq a \leq k} (1 - i_a) \left(\sum_{b < a} i_b \right) \end{aligned}$$

Proof. The proof is along the lines of the previous theorems. Use proposition 1.9, 1.10 to lift h to a map H , f to F and g to G . In that setting you can apply the rule for homotopy, project the formula to the first component and lastly you recognize the sign that comes from the shifts. \square

2. THE CONFIGURATION SPACES $Conf_n(\mathbb{R})$, $C_n(\mathbb{R})$ AND $\mathfrak{C}_n(\mathbb{R})$

2.1. Families of uncompactified configuration spaces. Given a set A we define the configuration space $Conf_A(\mathbb{R})$ as the set of injections of the set A into the real line;

$$Conf_A(\mathbb{R}) := \{A \hookrightarrow \mathbb{R}\}.$$

We think of this as $|A|$ distinct real points labeled by the elements from A . In the special case when $A = [n]$ we use the notation

$$Conf_n(\mathbb{R}) := Conf_{[n]}(\mathbb{R}).$$

Sometimes we will consider the full set of maps $A \rightarrow \mathbb{R}$, and for it we introduce the notation

$$\widetilde{Conf_A}(\mathbb{R}) := \{A \rightarrow \mathbb{R}\}.$$

The set $Conf_n(\mathbb{R})$ is a real oriented manifold of dimension n . As a space $Conf_n(\mathbb{R})$ is the union of $n!$ connected components, all isomorphic to

$$Conf_n^o(\mathbb{R}) := \{x_1 < x_2 < \dots < x_n\}.$$

The orientation is given on $Conf_n^o(\mathbb{R})$ as the volume form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$. The group S_n acts on $Conf_n(\mathbb{R})$ by permuting the elements of $[n]$. We assume that the action of S_n is orientation preserving on $Conf_n(\mathbb{R})$ and this fixes the orientation on all connected components of $Conf_n(\mathbb{R})$.

The 2-dimensional Lie group $G_{(2)} = \mathbb{R}^+ \ltimes \mathbb{R}$ acts freely on $Conf_n(\mathbb{R})$ via the action

$$(x_1, \dots, x_n) \times (\lambda, \nu) = (\lambda x_1 + \nu, \dots, \lambda x_n + \nu).$$

The quotient space from this action is a $(n-2)$ -dimensional real oriented manifold. Where we choose to represent equivalence classes of this quotient space with elements of the form $(0 = x_1 < x_2 < \dots < x_{n-1} < x_n = 1)$, when we do so we will use the notation $C_n^o(\mathbb{R})$;

$$C_n^o(\mathbb{R}) := \{(0 = x_1 < x_2 < \dots < x_{n-1} < x_n = 1)\}.$$

The orientation is given by the form $dx_2 \wedge \dots \wedge dx_{n-1}$. We also define $C_n(\mathbb{R}) := S_n \times C_n^o(\mathbb{R})$.

Alternatively we can represent equivalence classes of

$$Conf_n(\mathbb{R})/G_{(2)}$$

with elements $p = (x_1, \dots, x_n) \in Conf_n(\mathbb{R})$ subject to

$$x_c(p) = \frac{1}{n} \sum x_i = 0 \quad ||p|| = \sqrt{\sum (x_i - x_c(p))^2} = 1.$$

We define

$$C_n^{st}(\mathbb{R}) := \{p \in Conf_n(\mathbb{R}) | x_c(p) = 0, ||p|| = 1\}$$

and also

$$\widetilde{C_n^{st}}(\mathbb{R}) := \{p \in \widetilde{Conf_n}(\mathbb{R}) | x_c(p) = 0, ||p|| = 1\}.$$

The 1-dimensional Lie group $G_{(1)} = \mathbb{R}$ acts on $Conf_n(\mathbb{R})$ by translation

$$(p, \nu) \mapsto p + \nu,$$

and we denote the quotient space

$$\mathfrak{C}_n(\mathbb{R}) := Conf_n(\mathbb{R})/G_{(1)}.$$

Define

$$\mathfrak{C}_n^{st}(\mathbb{R}) := \{p \in Conf_n(\mathbb{R}) | x_c(p) = 0\} \subset Conf_n(\mathbb{R}).$$

We have three isomorphisms associated to these configuration spaces.

- (1) The space $C_n(\mathbb{R})$ is naturally isomorphic to $C_n^{st}(\mathbb{R})$
- (2) We have

$$\Psi_n : \mathfrak{C}_n(\mathbb{R}) \cong C_n^{st}(\mathbb{R}) \times (0, 1)$$

given by

$$p \mapsto \left(\frac{p - x_c(p)}{\|p\|}, \frac{\|p\|}{1 + \|p\|} \right).$$

- (3) We have

$$\Phi_n : Conf_n(\mathbb{R}) \cong C_n^{st}(\mathbb{R}) \times (0, 1) \times (-1, 1)$$

given by

$$p \mapsto \left(\frac{p - x_c(p)}{\|p\|}, \frac{\|p\|}{1 + \|p\|}, \frac{x_c(p)}{1 + |x_c(p)|} \right).$$

These isomorphisms open up the door for compactifications of these configuration spaces which we discuss next.

2.2. A compactification of $C_n(\mathbb{R})$. We introduce a topological compactification $\overline{C}_n(\mathbb{R})$ as the closure of the following injections

$$C_n(\mathbb{R}) \xrightarrow{\Pi \pi_A} \prod_{|A| \subset [n], |A| \geq 2} C_A(\mathbb{R}) \xrightarrow{\simeq} \prod_{|A| \subset [n], |A| \geq 2} C_A^{st}(\mathbb{R}) \hookrightarrow \prod_{|A| \subset [n], |A| \geq 2} \tilde{C}_A^{st}(\mathbb{R})$$

The codimension one boundary strata of the configurations space $\overline{C}_n(\mathbb{R})$ is given by

•

$$\partial \overline{C}_n(\mathbb{R}) = \bigcup_{A \subset [n]} \overline{C}_{n-|A|+1}(\mathbb{R}) \times \overline{C}_{|A|}(\mathbb{R}),$$

where A is a connected proper subset of $[n]$ with two or more elements.

- The face complex on $\overline{C}_\bullet(\mathbb{R})$ has the natural structure of a dg free operad;

$$\mathcal{F}ree \left\langle \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{q-1} \quad i_q \end{array} \right\rangle_{q \geq 2}$$

where the differential acts as follows

$$\partial \left(\begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{q-1} \quad i_q \end{array} \right) = \sum_{k=1}^{q-2} \sum_{l=2}^{n-k} \epsilon(k, l) \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ i_1 \quad \dots \quad i_k \quad \bullet \quad i_{k+l+1} \quad \dots \quad i_q \\ \swarrow \quad \downarrow \quad \searrow \\ i_{k+1} \quad \dots \quad i_{k+l} \end{array}$$

Where the factor $\epsilon(k, l)$ is a sign that can be worked out to be $(-1)^{k+l(n-k-l)+1}$. Representations of this operad in differential graded vector space are given by A_∞ structures. Thus this is a description of the A_∞ operad.

2.3. The space $\overline{C}_n(\mathbb{R})$ as a smooth manifold with corners. Let $RT_{n,l}$ be the set of rooted trees with n legs and $l+1$ internal vertices. The set $RT_{n,l}$ parametrizes the codimension l boundary strata of $\overline{C}_n(\mathbb{R})$ in the following sense. Each tree $t \in RT_{n,l}$ describes a space $C_t(\mathbb{R})$ which is defined as the product

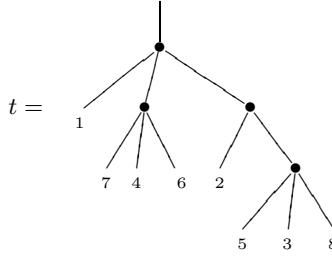
$$C_t(\mathbb{R}) := \prod_{v \in \text{vert}(t)} C_{|in(v)|}(\mathbb{R}),$$

where, like before, $\text{vert}(t)$ denote the set of internal vertices of t and $in(v)$ the set of input edges at the vertex v . From this one gets a description of $\overline{C}_n(\mathbb{R})$ as a stratified disjoint union of spaces

$$\overline{C}_n(\mathbb{R}) = \coprod_{l \geq 0} \coprod_{t \in RT_{n,l}} C_t(\mathbb{R}).$$

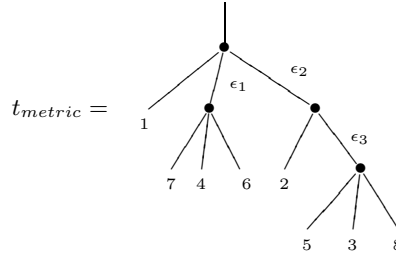
To make the compactified configuration space $\overline{C}_n(\mathbb{R})$ into a smooth manifold with corner we shall define coordinate charts U_t near the boundary stratum $C_t(\mathbb{R})$. We do this for a specific tree t but the general procedure

should be clear from the given example. Let t be the tree



We define the coordinate chart close to $C_t(\mathbb{R})$ in a three step procedure.

- (1) Associate to the tree t a *metric tree*, t_{metric} by endowing each internal edge with a bounded non-negative parameter ϵ ;



with $\epsilon_1, \epsilon_2, \epsilon_3 \in [0, \epsilon]$.

- (2) Pick an S_n -equivariant section $\gamma : C_n(\mathbb{R}) \rightarrow Conf_n(\mathbb{R})$, of the natural projection $Conf_n(\mathbb{R}) \rightarrow C_n(\mathbb{R})$ and associate to the image of γ a smooth structure. The section could be either of the two description of $C_n(\mathbb{R})$ we mentioned above; $C_n^{st}(\mathbb{R})$ or the space of configurations where $x_1 = 0$ and $x_n = 1$.
- (3) The coordinate chart U_t can now be seen to be isomorphic to the smooth manifold with corners $[0, \epsilon]^{|E(t)|} \times \prod_{v \in vert(t)} C_{|in(v)|}(\mathbb{R})$. The isomorphism is given by the map Φ_t ,

$$\Phi_t : [0, \epsilon]^{|E(t)|} \times \prod_{v \in vert(t)} C_{|in(v)|}(\mathbb{R}) \rightarrow U_t.$$

which we describe in the example of our tree t . Coordinatewise it is defined as follows

$$\begin{aligned} (0, \epsilon)^3 \times C_3^{st}(\mathbb{R}) \times C_3^{st}(\mathbb{R}) \times C_2^{st}(\mathbb{R}) \times C_3^{st}(\mathbb{R}) &\longrightarrow C_8(\mathbb{R}) \\ (\epsilon_1, \epsilon_2, \epsilon_3) \times (x_1, x', x'') \times (x_7, x_4, x_6) \times (x_2, x''') \times (x_5, x_3, x_8) &\longrightarrow (y_1, y_7, y_4, y_6, y_2, y_5, y_3, y_8) \end{aligned}$$

according to

$$\begin{aligned} y_1 &= x_1 & y_2 &= x'' + \epsilon_2 x_2 \\ y_3 &= x'' + \epsilon_2(x''' + \epsilon_3 x_3) & y_4 &= x' + \epsilon_1 x_4 \\ y_5 &= x'' + \epsilon_2(x''' + \epsilon_3 x_5) & y_6 &= x' + \epsilon_1 x_6 \\ y_7 &= x' + \epsilon_1 x_4 & y_8 &= x'' + \epsilon_2(x''' + \epsilon_3 x_5) \end{aligned}$$

In general the map Φ_t is given as the recursive ϵ -magnified substitution scheme. If the coordinates x_i, \dots, x_{i+k} lie in a corolla controlled by the internal edge associated to the coordinate x' and where the internal edge is parametrized by the factor ϵ , then the substitution give the new coordinates $x' + \epsilon x_i, \dots, x' + \epsilon x_{i+k}$.

2.4. A compactification of $\widehat{\mathfrak{C}}_n(\mathbb{R})$. Define the compactification of $\mathfrak{C}_n(\mathbb{R})$ as the closure of the following inclusions

$$\mathfrak{C}_n(\mathbb{R}) \xrightarrow{\Pi \pi_A} \prod_{|A| \subset [n], |A| \geq 1} \mathfrak{C}_A(\mathbb{R}) \xrightarrow{\Pi \Phi_A} \prod_{|A| \subset [n], |A| \geq 1} C_A^{st}(\mathbb{R}) \times (0, 1) \hookrightarrow \prod_{|A| \subset [n], |A| \geq 1} \widetilde{C}_A^{st}(\mathbb{R}) \times [0, 1]$$

The codimension one boundary strata of the configurations space $\widehat{\mathfrak{C}}_n(\mathbb{R})$ is given by

•

$$\partial \widehat{\mathfrak{C}}_n(\mathbb{R}) = \bigcup \widehat{\mathfrak{C}}_{n-|A|+1}(\mathbb{R}) \times \overline{C}_{|A|}(\mathbb{R}) \cup \bigcup \overline{C}_k(\mathbb{R}) \times \widehat{\mathfrak{C}}_{|A_1|}(\mathbb{R}) \times \dots \times \widehat{\mathfrak{C}}_{|A_k|}(\mathbb{R})$$

where A is as above and where the A_i are connected disjoint subsets of $[n]$ such that $\inf A_1 < \dots < \inf A_k$ and $\cup A_i = [n]$.

- The face complex of the disjoint union

$$\overline{\mathcal{C}}_\bullet(\mathbb{R}) \sqcup \widehat{\mathcal{C}}_\bullet(\mathbb{R}) \sqcup \overline{\mathcal{C}}_\bullet(\mathbb{R})$$

has the natural structure of a dg free operad of transformation type;

$$\mathcal{F}ree \left\langle \begin{array}{c} \text{---} \bullet \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{q-1} \quad i_q \end{array}, \begin{array}{c} \text{---} \blacksquare \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{n-1} \quad i_n \end{array}, \begin{array}{c} \text{---} \circ \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{n-1} \quad i_p \end{array} \right\rangle_{\substack{p, q \geq 2 \\ n \geq 1}}$$

The differential has the following action

$$\begin{aligned} \partial \left(\begin{array}{c} \text{---} \blacksquare \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{n-1} \quad i_n \end{array} \right) &= \sum_{l=1}^{n-1} \sum_{k=1}^{n-l} \epsilon(l, k) \begin{array}{c} \text{---} \blacksquare \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad \dots \quad i_k \quad \dots \quad i_{k+l+1} \quad \dots \quad i_n \\ \text{---} \circ \text{---} \\ \diagup \quad \diagdown \\ i_{k+1} \quad \dots \quad i_{k+l} \end{array} \\ &+ \sum_{k=2}^n \sum_{n=n_1+\dots+n_k} \epsilon(k; n_1, \dots, n_k) \begin{array}{c} \text{---} \blacksquare \text{---} \quad \text{---} \blacksquare \text{---} \quad \dots \quad \text{---} \blacksquare \text{---} \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \dots \quad \diagup \quad \diagdown \\ i_1 \quad \dots \quad i_{n_1} \quad i_{n_1+1} \quad \dots \quad i_{n_1+n_2} \quad \dots \quad i_n \end{array} \end{aligned}$$

Where $\epsilon(k, l) = (-1)^{k+l(n-k)+1}$ and

$$\epsilon(k; n_1, \dots, n_k) = (-1)^{(k-1)(n_1-1) + (k-2)(n_2-1) + \dots + 2(n_{k-2}-1) + n_{k-1} - 1}$$

On the corollas corresponding to the A_∞ structure,

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{q-1} \quad i_q \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \circ \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{n-1} \quad i_p \end{array},$$

the differential acts precisely like in the case $\overline{\mathcal{C}}_n(\mathbb{R})$. Representations of this operad are given by a three pieces of data: two A_∞ algebras, A and A' , and a morphism of A_∞ algebras $A \rightarrow A'$. Thus this is the previously discussed operad $\mathcal{M}or(As)_\infty$.

2.5. The space $\widehat{\mathcal{C}}_n(\mathbb{R})$ as a smooth manifold with corners. We generalize the procedure for $\overline{\mathcal{C}}_n(\mathbb{R})$ to $\widehat{\mathcal{C}}_n(\mathbb{R})$. For every tree $t \in \mathcal{M}or(A_\infty)$ we define the sets $vert_{\bullet, \circ}(t)$ and $vert_{\blacksquare}(t)$ as the vertices of t marked by $\{\bullet, \circ\}$ or \blacksquare . For the tree t we define $\mathcal{C}_t(\mathbb{R})$ as a product;

$$\mathcal{C}_t(\mathbb{R}) := \prod_{v \in vert_{\bullet, \circ}(t)} C_{|in(v)|}(\mathbb{R}) \times \prod_{v \in vert_{\blacksquare}(t)} \mathcal{C}_{|in(v)|}(\mathbb{R}).$$

We can describe the space $\widehat{\mathcal{C}}_n(\mathbb{R})$ as a stratified union of spaces;

$$\widehat{\mathcal{C}}_n(\mathbb{R}) = \prod_{t \in \mathcal{M}or(As)_\infty(n)} \mathcal{C}_t(\mathbb{R}).$$

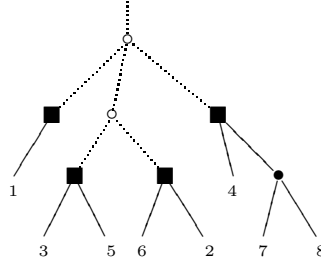
We shall define a coordinate chart U_t around every boundary stratum $\mathcal{C}_t(\mathbb{R})$ with a metric tree. We associate to t the metric tree t_{metric} with for

- (1) every internal edge of the type $\begin{array}{c} \blacksquare \\ | \\ \bullet \end{array}$ a small positive parameter ϵ ;
- (2) every vertex of a dashed corolla associate a large positive number τ ,

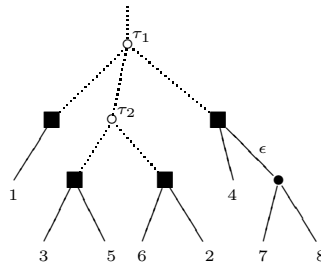
$$\begin{array}{c} \text{---} \circ \text{---} \\ \diagup \quad \diagdown \\ \dots \end{array}$$

- (3) every subgraph of t_{metric} of the type $\begin{array}{c} \circ^{\tau_1} \\ \vdots \\ \circ^{\tau_2} \end{array}$ an inequality $\tau_1 > \tau_2$.

Example 2.1. As an example we consider a specific tree. The general method should be clear from this description. Let t be the following tree



Then the associated metric tree, t_{metric} , is given by



Choose an equivariant section, $s : \mathfrak{C}_n(\mathbb{R}) \rightarrow \text{Conf}_n(\mathbb{R})$ to the projection $\text{Conf}_n(\mathbb{R}) \rightarrow \mathfrak{C}_n(\mathbb{R})$ and a smooth structure on the image of s . Define $\mathfrak{C}_n^{st}(\mathbb{R}) := s(\mathfrak{C}_n(\mathbb{R}))$, which is called the space of configurations in standard position. One possible choice of $\mathfrak{C}_n^{st}(\mathbb{R})$ is subspace of points in $\text{Conf}_n(\mathbb{R})$ where $\sum x_i = 0$.

The coordinate chart $U_t \subset \widehat{\mathfrak{C}}_n(\mathbb{R})$ is now defined to be isomorphic to the manifold with corners,

$$(l, +\infty]^{|vert_o|(t)} \times [0, s]^{|edge_{\blacksquare}(t)|} \times \prod_{v \in vert_{o, \bullet}(t)} C_{|in(v)|}^{st}(\mathbb{R}) \times \prod_{v \in vert_{\blacksquare}(t)} \mathfrak{C}_{|in(t)|}^{st}(\mathbb{R})$$

where $vert_o$ denotes the set of vertices of type \circ , $vert_{o, \bullet}$ denotes the set of vertices of type \circ or \bullet and $edge_{\blacksquare}$ denote the set of edges of type \blacksquare . The isomorphism Φ_t between the coordinate chart U_t and the product above is read from metric tree. We map is given in coordinates, for the specific tree in the above example, as follows

$$\begin{aligned} (l, +\infty]^2 &\times [0, s] \times C_3^{st}(\mathbb{R}) \times C_2^{st}(\mathbb{R}) \times C_2^{st}(\mathbb{R}) \\ (\tau_1, \tau_2) &\times \epsilon \times (x', x'', x''') \times (t', t'') \times (x_7, x_8) \\ &\times \mathfrak{C}_1^{st}(\mathbb{R}) \times \mathfrak{C}_2^{st}(\mathbb{R}) \times \mathfrak{C}_2^{st}(\mathbb{R}) \times \mathfrak{C}_2^{st}(\mathbb{R}) \\ &\times x_1 \times (x_3, x_5) \times (x_6, x_2) \times (x_4, u) \end{aligned} \longrightarrow \begin{aligned} &\mathfrak{C}_8(\mathbb{R}) \\ &(y_1, \dots, y_8) \end{aligned}$$

such that

$$\begin{aligned} y_1 &= \tau_1 x' + x_1 & y_2 &= \tau_1 x'' + \tau_2 t'' + x_2 \\ y_3 &= \tau_1 x'' + \tau_2 t' + x_3 & y_4 &= \tau_1 x''' + x_4 \\ y_5 &= \tau_1 x'' + \tau_2 t' + x_5 & y_6 &= \tau_1 x'' + \tau_2 t'' + x_6 \\ y_7 &= \tau_1 x''' + u + \epsilon x_7 & y_8 &= \tau_1 x''' + u + \epsilon x_8 \end{aligned}$$

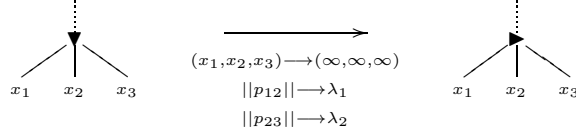
The boundary strata in U_t are given by allowing formally $\tau_1 = \infty, \tau_2 = \infty$ such that $\tau_1/\tau_2 = 0$ and $\epsilon = 0$.

3. THE OPERAD $\mathcal{Ho}(As)_\infty$

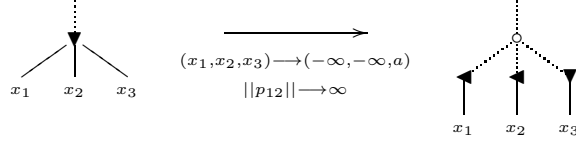
3.1. Compactification of the Configuration Space $\text{Conf}_\bullet(\mathbb{R})$. In this section we introduce our main result. We define the new compactification of the configuration space $\text{Conf}_n(\mathbb{R})$ as the closure of the following injections

$$\begin{aligned} \text{Conf}_n(\mathbb{R}) &\xrightarrow{\Pi \pi_A} \prod_{|A| \subset [n], |A| \geq 1} \text{Conf}_A(\mathbb{R}) \\ &\downarrow \Pi \Phi_A \\ \prod_{|A| \subset [n], |A| \geq 1} C_A^{st}(\mathbb{R}) \times (0, 1) \times (-1, 1)^c &\hookrightarrow \prod_{|A| \subset [n], |A| \geq 1} \tilde{C}_A^{st}(\mathbb{R}) \times [0, 1] \times [-1, 1] \end{aligned}$$

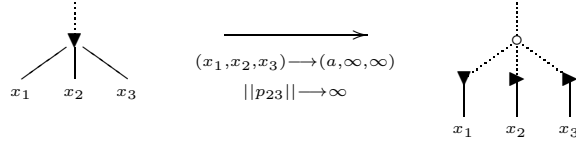
- (2) Analogous to above we can consider $p = (x_1, x_2, x_3) \rightarrow (\infty, \infty, \infty)$ in such a way that $\|p_{12}\| = \lambda_1$ and $\|p_{23}\| = \lambda_2$ remains constant. This boundary strata can also be identified with a copy of $\widehat{\mathfrak{C}}_3(\mathbb{R})$.



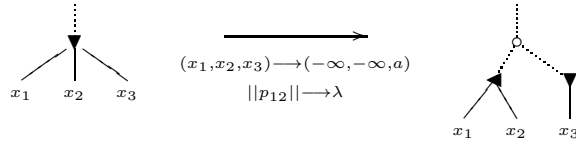
- (3) $p = (x_1, x_2, x_3) \rightarrow (-\infty, -\infty, a)$ in such a way that $\|p_{12}\|$ diverges. Points of this type can be identified with $\overline{\mathcal{C}}_3(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R}) \times \overline{\text{Conf}}_1(\mathbb{R})$.



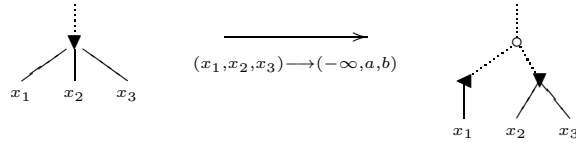
- (4) $p = (x_1, x_2, x_3) \rightarrow (a, \infty, \infty)$ in such a way that $\|p_{23}\|$ diverges. Points of this type can be identified with $\overline{\mathcal{C}}_3(\mathbb{R}) \times \overline{\text{Conf}}_1(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R})$.



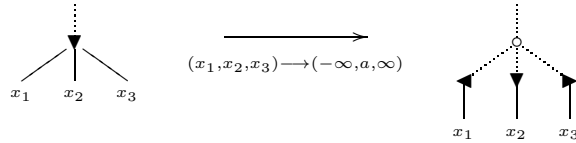
- (5) $p = (x_1, x_2, x_3) \rightarrow (-\infty, -\infty, a)$ in such a way that $\|p_{12}\|$ converges. Points of this type can be identified with $\overline{\mathcal{C}}_2(\mathbb{R}) \times \widehat{\mathfrak{C}}_2(\mathbb{R}) \times \overline{\text{Conf}}_1(\mathbb{R})$.



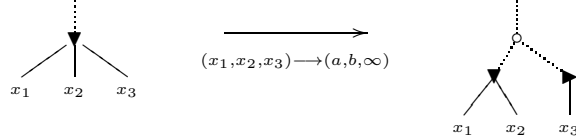
- (6) $p = (x_1, x_2, x_3) \rightarrow (-\infty, a, b)$ where $a < b$; which can be identified with $\overline{\mathcal{C}}_3(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R}) \times \overline{\text{Conf}}_2(\mathbb{R})$



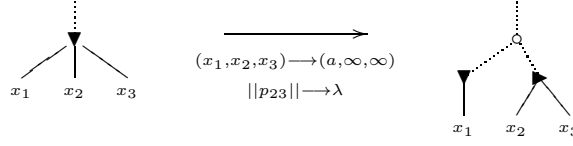
- (7) $p = (x_1, x_2, x_3) \rightarrow (-\infty, a, \infty)$. These points can be identified with $\overline{\mathcal{C}}_3(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R}) \times \overline{\text{Conf}}_1(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R})$



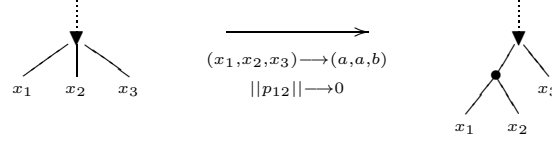
- (8) $p = (x_1, x_2, x_3) \rightarrow (a, b, \infty)$, where $a < b$. Points of this type can be identified with $\overline{\mathcal{C}}_2(\mathbb{R}) \times \overline{\text{Conf}}_2(\mathbb{R}) \times \widehat{\mathfrak{C}}_1(\mathbb{R})$.



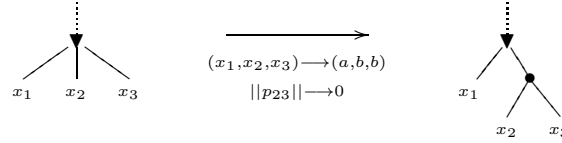
- (9) $p = (x_1, x_2, x_3) \rightarrow (a, \infty, \infty)$ in such a way that $\|p_{23}\|$ converges. This boundary strata can be identified with $\overline{\mathcal{C}_2}(\mathbb{R}) \times \overline{\text{Conf}_2}(\mathbb{R}) \times \widehat{\mathcal{C}_1}(\mathbb{R})$



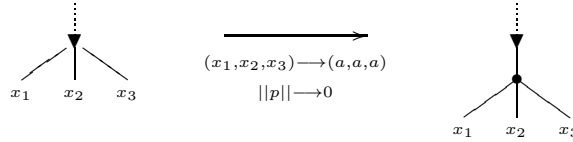
- (10) $p = (x_1, x_2, x_3) \rightarrow (a, a, b)$, where $a < b$. Points of this type can be identified with $\overline{\text{Conf}_2}(\mathbb{R}) \times \overline{\mathcal{C}_2}(\mathbb{R})$.



- (11) $p = (x_1, x_2, x_3) \rightarrow (a, b, b)$, where $a < b$. This boundary strata can be identified with $\overline{\text{Conf}_2}(\mathbb{R}) \times \overline{\mathcal{C}_2}(\mathbb{R})$



- (12) $p = (x_1, x_2, x_3) \rightarrow (a, a, a)$. This boundary strata can be identified with $\overline{\text{Conf}_1}(\mathbb{R}) \times \overline{\mathcal{C}_3}(\mathbb{R})$



We summarize this in the formula:

$$\begin{aligned} \partial \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} &= - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} \\ &+ \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} \\ &+ \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ x_1 \quad x_2 \quad x_3 \end{array} \end{aligned}$$

We summarize the result in our main theorem

Theorem 3.3. The face complex on the disjoint union $\overline{\mathcal{C}_\bullet}(\mathbb{R}) \sqcup \widehat{\mathcal{C}_\bullet}(\mathbb{R}) \sqcup \overline{\text{Conf}_\bullet}(\mathbb{R}) \sqcup \widehat{\mathcal{C}_\bullet}(\mathbb{R}) \sqcup \overline{\mathcal{C}_\bullet}(\mathbb{R})$ is naturally a dg free operad of transformation type

$$\mathcal{H}o(As)_\infty := \text{Free} \left\langle \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{q-1} \quad i_q \end{array}, \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{k-1} \quad i_k \end{array}, \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{m-1} \quad i_m \end{array}, \right. \\ \left. \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{n-1} \quad i_n \end{array}, \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_{p-1} \quad i_p \end{array} \right\rangle_{p, q \geq 2, k, m, n \geq 1}.$$

Representation of this operad in a pair of vector spaces V^1 and V^2 is the structure of two A_∞ algebras, (V^1, μ^1) and (V^2, μ^2) , two A_∞ morphisms, $f, g : (V^1, \mu^1) \rightarrow (V^2, \mu^2)$ and a homotopy h between the morphism $h : f \rightarrow g$. The action of the differential was described earlier.

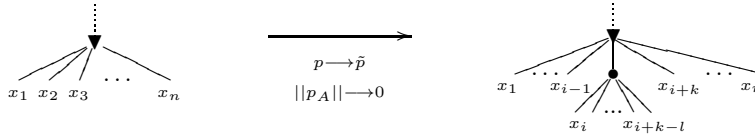
Proof. The proof is by inspection. We have worked out the cases of two and three points in detail and we can see that they correspond to the algebraic formulas of the previous section. The general case is treated in complete analogy.

Let p be a configuration of n points on the real line $p = (x_1 < x_2 < \dots < x_n) \in \text{Conf}_n(\mathbb{R})$. The possible codimension 1 boundary strata can arise in three different ways.

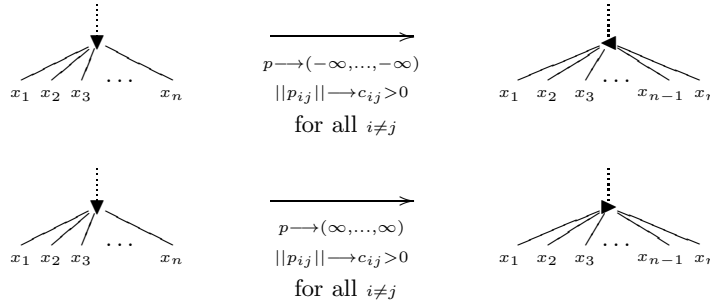
- (1) A connected subset $A = (x_i < x_{i+1} < \dots < x_{i+k-1})$ of points collapsing into single point; A limit point

$$p \longrightarrow \tilde{p} = (a_1 < a_2 < \dots < a_{i-1} < a_i = a_{i+1} = \dots = a_{i+k-1} < a_{i+k} < \dots < a_n).$$

Points of this type can be identified with $\overline{\text{Conf}}_{n-k+1}(\mathbb{R}) \times \overline{\mathcal{C}}_k(\mathbb{R})$.

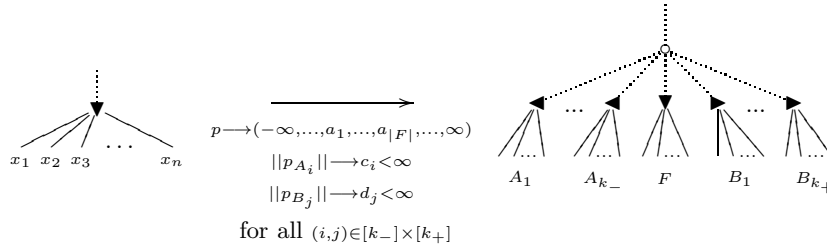


- (2) All n points moving in a cluster towards $\pm\infty$; A limit point $p \longrightarrow \pm(\infty, \infty, \dots, \infty)$ where the distance between points remain finite, e.g. it could look like $p = (t + \lambda_1, t + \lambda_2, \dots, t + \lambda_n)$ with $\lambda_1 < \lambda_2 < \dots < \lambda_n$ and $t \longrightarrow \pm\infty$. Limit points of this type can be identified with $\widehat{\mathcal{C}}_n(\mathbb{R})$.



- (3) For each $k \geq 2$ the n points converge to $k = k_- + 1 + k_+$ clusters where k_- clusters move to $-\infty$, k_+ clusters move to $+\infty$ and one cluster where each point converge to a finite point. Within each of the $k_- + k_+$ clusters moving to $\pm\infty$ the distance between points remain finite, while the distance from any two points from different clusters tend to ∞ . Every such configuration is determined by a disjoint union of connected subsets $A_1 \cup \dots \cup A_{k_-} \cup F \cup B_1 \cup \dots \cup B_{k_+} = [n]$ is with $\inf A_1 < \inf A_2 < \dots < \inf A_{k_-} < \inf F < \inf B_1 < \inf B_2 < \dots < \inf B_{k_+}$, and limit points of this type can then be identified with

$$\overline{\mathcal{C}}_k(\mathbb{R}) \times \widehat{\mathcal{C}}_{|A_1|}(\mathbb{R}) \times \dots \times \widehat{\mathcal{C}}_{|A_{k_-}|}(\mathbb{R}) \times \overline{\text{Conf}}_{|F|}(\mathbb{R}) \times \widehat{\mathcal{C}}_{|B_1|}(\mathbb{R}) \times \dots \times \widehat{\mathcal{C}}_{|B_{k_+}|}(\mathbb{R}).$$



□

3.2. The space $\overline{\text{Conf}}_n(\mathbb{R})$ as a smooth manifold with corners. We shall endow the space $\overline{\text{Conf}}_n(\mathbb{R})$ with a manifold structure in an almost identical procedure to how the space $\widehat{\mathcal{C}}_n(\mathbb{R})$ was treated. For every tree $t \in \mathcal{Ho}(As)_\infty$ we define the sets $\text{vert}_{\bullet, \circ}(t)$, $\text{vert}_{\blacktriangleleft, \blacktriangleright}(t)$ and $\text{vert}_{\blacktriangledown}$ as the vertices of t marked by $\{\bullet, \circ\}$, $\{\blacktriangleleft, \blacktriangleright\}$ or \blacktriangledown , respectively. For the tree t we define $\text{Conf}_t(\mathbb{R})$ as a product;

$$\text{Conf}_t(\mathbb{R}) := \prod_{v \in \text{vert}_{\bullet, \circ}(t)} \mathcal{C}_{|in(v)|}(\mathbb{R}) \times \prod_{v \in \text{vert}_{\blacktriangleleft, \blacktriangleright}(t)} \mathcal{C}_{|in(v)|}(\mathbb{R}) \times \prod_{v \in \text{vert}_{\blacktriangledown}} \text{Conf}_n(\mathbb{R}).$$

We can describe the space $\overline{\text{Conf}}_n(\mathbb{R})$ as a stratified union of spaces;

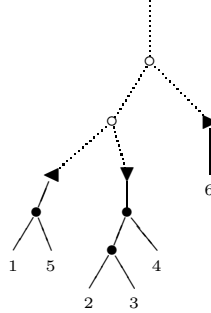
$$\overline{\text{Conf}}_n(\mathbb{R}) = \prod_{t \in \mathcal{Ho}(As)_\infty(n)} \text{Conf}_t(\mathbb{R}).$$

We shall define a coordinate chart U_t around every boundary stratum $Conf_t(\mathbb{R})$ with a metric tree. We associate to t the metric tree t_{metric} with for

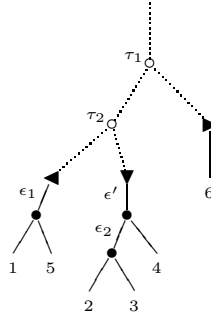
- (1) every internal edge of the types $\begin{smallmatrix} \uparrow \\ \bullet \end{smallmatrix}$, $\begin{smallmatrix} \uparrow \\ \circ \end{smallmatrix}$ or $\begin{smallmatrix} \uparrow \\ \blacktriangledown \end{smallmatrix}$ a small positive parameter ϵ ;
- (2) every vertex of a dashed corolla associate a large positive number τ ,

- (3) every subgraph of t_{metric} of the type $\begin{smallmatrix} \circ \\ \tau_1 \\ \tau_2 \end{smallmatrix}$ an inequality $\tau_1 > \tau_2$.

Example 3.4. We consider a specific tree and associate the metric tree to it. The general method should be clear from this description. Let t be the following tree



Then the associated metric tree, t_{metric} , is given by



The coordinate chart $U_t \subset \overline{Conf}_n(\mathbb{R})$ is now defined to be isomorphic to the manifold with corners,

$$(l, +\infty]^{|vert_o(t)|} \times [0, s]^{|edge_{\begin{smallmatrix} \uparrow \\ \bullet \end{smallmatrix}, \begin{smallmatrix} \uparrow \\ \circ \end{smallmatrix}, \begin{smallmatrix} \uparrow \\ \blacktriangledown \end{smallmatrix}}(t)|} \times \prod_{v \in vert_{\circ, \bullet}(t)} C_{|in(v)|}^{st}(\mathbb{R}) \\ \times \prod_{v \in vert_{\blacktriangleleft, \blacktriangleright}(t)} \mathfrak{C}_{|in(t)|}^{st}(\mathbb{R}) \times \prod_{v \in vert_{\blacktriangledown}(t)} Conf_{|in(v)|}(\mathbb{R})$$

where $vert_o$ denotes the set of vertices of type \circ , $vert_{\circ, \bullet}$ denotes the set of vertices of type \circ or \bullet and so forth. The set $edge_{\begin{smallmatrix} \uparrow \\ \bullet \end{smallmatrix}, \begin{smallmatrix} \uparrow \\ \circ \end{smallmatrix}, \begin{smallmatrix} \uparrow \\ \blacktriangledown \end{smallmatrix}}$ is give set of edges of type $\begin{smallmatrix} \uparrow \\ \bullet \end{smallmatrix}$, $\begin{smallmatrix} \uparrow \\ \circ \end{smallmatrix}$ or $\begin{smallmatrix} \uparrow \\ \blacktriangledown \end{smallmatrix}$. The isomorphism Φ_t between the coordinate chart U_t and the product above is read from the metric tree. The map is given in coordinates, for the specific tree in the above example, as follows

$$\begin{array}{ccccccccc} (l, +\infty]^2 & \times & [0, s]^3 & \times & C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) \\ (\tau_1, \tau_2) & \times & (\epsilon_1, \epsilon_2, \epsilon') & \times & (x'_1, x'_2) & \times & (x'_1, x'_2) & \times & (x_1, x_5) \\ \\ C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) & \times & \mathfrak{C}_1^{st}(\mathbb{R}) & \times & \mathfrak{C}_1^{st}(\mathbb{R}) & \times & Conf_1(\mathbb{R}) \\ (x', x_4) & \times & (x_2, x_3) & \times & x_6 & \times & s & \times & u \\ & & & & & & & & \longrightarrow Conf_6(\mathbb{R}) \\ & & & & & & & & (y_1, \dots, y_6) \end{array}$$

such that

$$\begin{array}{ll} y_1 = & \tau_1 x'_1 + \tau_2 x_1 + t + \epsilon_1 x_1 & y_2 = & \tau_1 x'' + \tau_2 x' + u + \epsilon'(x' + \epsilon_2 x_2) \\ y_3 = & \tau_1 x'' + \tau_2 x' + u + \epsilon'(x' + \epsilon_2 x_3) & y_4 = & \tau_1 x'' + \tau_2 x' + u + \epsilon' x_4 \\ y_5 = & \tau_1 x'_1 + \tau_2 x_1 + t + \epsilon_1 x_5 & y_6 = & \tau_1 x'' + x_6 \end{array}$$

The boundary strata in U_t are given by allowing formally $\tau_1 = \infty, \tau_2 = \infty$ such that $\tau_1/\tau_2 = 0$ and $\epsilon_1 = 0, \epsilon_2 = 0, \epsilon' = 0$.

3.3. The Cohomology of $\mathcal{H}o(As)_\infty$. We will need two results in order to calculate the cohomology of the operad $\mathcal{H}o(As)_\infty$.

Theorem 3.5. *Let P be a Koszul operad. Define the two colored operad $\mathcal{M}or(P)$ whose representations are two P -algebras and a P -algebra morphism between them. The operad $\mathcal{M}or(P)$ has a minimal model given by the operad $\mathcal{M}or(P)_\infty$ whose representations are two P_∞ -algebras and a homotopy P_∞ -morphism between them.*

Proof. See [MV09a],[MV09b]. \square

Corollary 3.6. *The operad $\mathcal{M}or(As)$ has a minimal model given by $\mathcal{M}or(As)_\infty$.*

Lemma 3.7. *Let $f : B \rightarrow C$ be a map of filtered complexes, where both B and C are complete and exhaustive. Fix $r \geq 0$. Suppose that $f^r : E_{pq}^r(B) \cong E_{pq}^r(C)$ for all p and q . Then $f : H(B) \rightarrow H(C)$ is an isomorphism.*

This result is known as the comparison lemma, and can be found in a textbook on homological algebra, e.g. [We].

We can now state our result.

Theorem 3.8. *The cohomology of $\mathcal{H}o(As)_\infty$ is the operad $\mathcal{M}or(As)$ whose representations are a pair of associative algebras and a morphism of associative algebras between them.*

Proof. There is a natural projection of operads

$$\pi : \mathcal{H}o(As)_\infty \twoheadrightarrow \mathcal{M}or(As)_\infty.$$

We can describe this map on corollas by using the presentation of $\mathcal{H}o(As)_\infty$ and $\mathcal{M}or(As)_\infty$ from theorem 3.3 and section 2.4, respectively;

$$\begin{aligned} \pi \left(\begin{array}{c} \downarrow \\ \diagup \quad \diagdown \\ \vdots \end{array} \right) &= 0 \\ \pi \left(\begin{array}{c} \downarrow \\ \diagup \quad \blacktriangle \quad \diagdown \\ \vdots \end{array} \right) &= \pi \left(\begin{array}{c} \downarrow \\ \diagup \quad \blacktriangleleft \quad \diagdown \\ \vdots \end{array} \right) = \begin{array}{c} \downarrow \\ \diagup \quad \blacksquare \quad \diagdown \\ \vdots \end{array} \\ \pi \left(\begin{array}{c} \downarrow \\ \diagup \quad \bullet \quad \diagdown \\ \vdots \end{array} \right) &= \begin{array}{c} \downarrow \\ \diagup \quad \bullet \quad \diagdown \\ \vdots \end{array} \\ \pi \left(\begin{array}{c} \downarrow \\ \diagup \quad \circ \quad \diagdown \\ \vdots \end{array} \right) &= \begin{array}{c} \downarrow \\ \diagup \quad \circ \quad \diagdown \\ \vdots \end{array} \end{aligned}$$

The map π obviously respects the differentials of the operads.

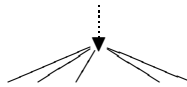
We introduce a filtration on $\mathcal{H}o(As)_\infty(n)$ and $\mathcal{M}or(As)_\infty(n)$ on the number of internal vertices in a tree,

$$F_P \mathcal{H}o(As)_\infty(n) = \{x \in \mathcal{H}o(As)_\infty(n) \mid \text{number of internal vertices of } x \geq p\}$$

and

$$F_P \mathcal{M}or(As)_\infty(n) = \{x \in \mathcal{M}or(As)_\infty(n) \mid \text{number of internal vertices of } x \geq p\}.$$

Clearly the differentials in $\mathcal{H}o(As)_\infty$ and $\mathcal{M}or(As)_\infty$ respect these filtrations as the number of vertices can only stay the same or increase when the differentials are applied. Note that the filtrations are both exhaustive and complete, this follows from that the objects in question are finite dimensional for any given n . The induced differential on $E_{pq}^0(\mathcal{M}or(As)_\infty)$ will either map a corolla to zero or increase the number of vertices and therefore $E_{pq}^1(\mathcal{M}or(As)_\infty) = H(E_{pq}^0(\mathcal{M}or(As)_\infty)) = E_{pq}^0(\mathcal{M}or(As)_\infty)$. On the other hand, in the case of $E_{pq}^0(\mathcal{H}o(As)_\infty)$, we have that the differential will map all trees except those containing a corolla of type



to zero. We get that the image of $\partial^0 : E_{pq}^0(\mathcal{H}o(As)_\infty) \rightarrow E_{pq}^0(\mathcal{H}o(As)_\infty)$ will consist of trees (operadically) generated by the difference of corollas;

$$\left(\begin{array}{c} \downarrow \\ \diagup \quad \blacktriangle \quad \diagdown \\ \vdots \end{array} \right) - \left(\begin{array}{c} \downarrow \\ \diagup \quad \blacktriangleleft \quad \diagdown \\ \vdots \end{array} \right) \in \mathcal{H}o(As)_\infty.$$

The first page is then determined;

$$\begin{aligned}
& E_{pq}^1(\mathcal{H}o(As)_\infty) \\
&= \text{Free} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) \\
&\cong E_{pq}^0(\mathcal{M}or(As)_\infty) \\
&= E_{pq}^1(\mathcal{M}or(As)_\infty)
\end{aligned}$$

By the comparison lemma 3.7 we find that the cohomology of $\mathcal{H}o(As)_\infty$ is the same as that of $\mathcal{M}or(As)_\infty$, which by the lemma 3.6 is precisely $\mathcal{M}or(As)$. \square

Corollary 3.9. *The operad $\mathcal{H}o(As)_\infty$ is a non-minimal quasi-free model of $\mathcal{H}o(As)$.*

Proof. There is a natural projection of operads

$$p : \mathcal{H}o(As)_\infty \twoheadrightarrow \mathcal{H}o(As).$$

We determine the cohomology of the operad $\mathcal{H}o(As)$. Let $H(\mathcal{H}o(As)) = Z/B$, then if μ_V and μ_W are the multiplications, $f, g : V \rightarrow W$ are the algebra morphisms and $h : f \sim g$ is the homotopy between them. We will have that $\partial f = \partial g = \partial \mu_V = \partial \mu_W = 0$, so the generators all constitute cycles. The boundaries are generated by $\partial h = f - g$. Hence

$$H(\mathcal{H}o(As)) = Z/B = \langle f, g, \mu_V, \mu_W \rangle / (f - g) \cong \langle [f], \mu_V, \mu_W \rangle$$

and we see that the cohomology is equal to $\mathcal{M}or(As)$. By the previous theorem the corollary is now implied. \square

Corollary 3.10. *The operad $\mathcal{H}o(As)_\infty$ is a non-minimal quasi-free model of $\mathcal{M}or(As)$.*

Proof. The model-structure comes from the map

$$\tilde{p} : \mathcal{H}o(As)_\infty \twoheadrightarrow \mathcal{M}or(As),$$

which is given by post-composing the map p from corollary 3.9 with the natural projection onto cohomology classes;

$$\mathcal{H}o(As) \twoheadrightarrow H(\mathcal{H}o(As)) \cong \mathcal{M}or(As).$$

\square

REFERENCES

- [BV] J.M. Boardman and R. M. Vogt *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Mathematics Number 347 (1973)
- [Co] F.R. Cohen, *The homology of C_{n+1} -spaces*, Lecture Notes in Mathematics 533, 207-351
- [GK] Victor Ginzburg and Mikhail Kapranov *Koszul duality for operads* Duke Math. J., 76(1):203–272, 1994.
- [Ka] T. V. Kadeishvili. *The algebraic structure in the homology of an $A(\infty)$ -algebra*. Soobshch. Akad. Nauk Gruzin. SSR, 108(2):249–252 (1983), 1982.
- [Ko1] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), 157-216.
- [Ko2] M. Kontsevich, *Operads and Motives in Deformation quantization*, Lett. Math. Phys. **48** (1999), 35-72.
- [Mar1] M. Markl, *Models for operads*, Commun. in Algebra **24** (1996), no. 4, 1471–1500
- [Mar2] M. Markl *Homotopy Algebras via Resolutions of Operads*, The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999). Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 157-164.
- [Mar3] M. Markl, *Loop Homotopy Algebras in Closed String Field Theory*, Communications in Mathematical Physics **221** (2001), no. 2,
- [MaShSt] M. Markl, S. Shnider and J.D. Stasheff, *Operads in Algebra, Topology and Physics* volume 96 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, Rhode Island, 2002.
- [Ma] P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, Volume 271 (1972)
- [Me] S. Merkulov *Operads, configuration spaces and quantization*, Bull. Braz. Math. Soc. (N.S.) 42 (2011), no.4, 683-781.
- [MV09a] S. Merkulov and B. Vallette. *Deformation theory of representations of prop(erad)s. I*. J. Reine Angew. Math., 634:51–106, 2009.
- [MV09b] S. Merkulov and B. Vallette. *Deformation theory of representations of prop(erad)s. II*. J. Reine Angew. Math., 636:123–174, 2009.
- [Sta63] J. D. Stasheff, *Homotopy associativity of H-spaces I & II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.
- [Tr] T. Tradler, *Infinity-Inner-Products on A-Infinity-Algebras* preprint arXiv:math/0108027
- [Ta] Dmitry E. Tamarkin. *Formality of chain operad of little discs* Lett. Math. Phys., 66(1-2):65–72, 2003.
- [VdL] P. van der Laan, *Coloured Koszul duality and strongly homotopy operads*, arXiv:math.QA/0312147 (2003).
- [We] C. A. Weibel *An introduction to homological algebra*, Cambridge studies in advanced mathematics 38, 1994
- [Zw] B. Zwiebach. *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation*. Nuclear Physics, B390:33–152, 1993.

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, 106 91, STOCKHOLM, SWEDEN
E-mail address: theob@math.su.se